A wide-band integral equation solution for EM scattering by thin sheets

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Summary

We present a numerical solution for analyzing the thin sheet electromagnetic (EM) scattering using the integral equation method. The frequency range is from 10^{-3} Hz to 100 MHz, with emphasis on the high-frequency applications. In principle there is no limit in the number of sheets used for the modeling purpose. For demonstration purpose, however, the number of sheets is limited to two. These sheets are characterized by different geometrical as well as electrical properties, and they are embedded in a three-layered earth of arbitrary electrical properties. We demonstrate the accuracy of our numerical solution through the comparison with the layered earth responses.

Introduction

The thin sheet modeling scheme has been an efficient tool that offers valuable insights into the 3-D EM scattering problems. Since Price (1949) introduced the innovative approach to this problem, various authors (Anan, 1974; Lajoie and West, 1976; Weidelt, 1981; Walker and West, 1991; Fainberg, et al., 1993) have presented numerical solutions to this problem.

The thin sheet approach is valid for modeling plate-like conductors. The thickness of the conductor should be electrically thin (Joshi et al., 1988) compared to the length and width. The most useful target using the thin sheet approach would be a fracture, or a system of fractures, a commonly encountered geologic feature in environmental and engineering problems. Since the thickness of the fracture is usually negligible the scattering current can only exist on the sheet with its polarization parallel to the sheet. This dramatically reduces the memory requirement and, as a result, it is possible to compute the EM responses of this important class of models even on a PC. However, theoretical formulation involved in the thin sheet problem is no less simple than that of the full 3-D integral equation.

Among other algorithms, Weidelt (1981) developed an elegant formulation for solving the thin sheet integral equation. We followed the same general approach, except that now the algorithm has been extended to include high frequency EM scattering problems in which the effect of the displacement current needs to be considered. Some formulations have been newly derived, and an analytic approach to the singular cell evaluation has been developed. Formulations given here are the generalization of the tensor representation of source potential and Green's function described by Weidelt.

Formulation of the thin sheet integral equation

We consider two sheets in a three-layered earth as shown in Figure 1. The source field can be generated by a remote or local electric or magnetic dipole source on the surface or in a borehole. The conductivity and electrical permittivity of host rock and the sheet models can be arbitrary. They can even be dispersive to accommodate electrical properties at high frequencies. A Cole-Cole (1941) relaxation formula, for example, can be used to describe the model parameters.

In the following formulation we only consider one thin sheet. Extension of this formula to include multiple sheets is trivial.

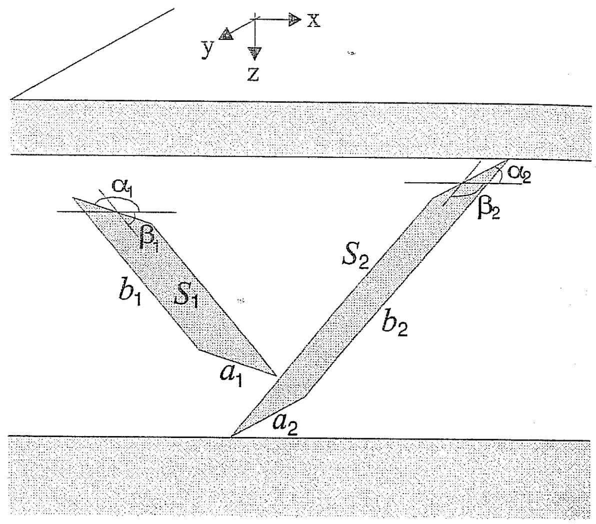

Fig. 1 Two thin sheets in a three-layered earth. a_i, b_i, α_i, and β_i are the strike length, depth extent in the dip direction, strike and dip angle of the i-th sheet, respectively.

It can be shown (Weidelt, 1981) that the electric field satisfies the integral expression

$$\mathbf{E}(\mathbf{r}_0) = \mathbf{E}^P(\mathbf{r}_0) - i\omega\mu \int_s \underline{\underline{\mathbf{G}}}(\mathbf{r}_0,\mathbf{r}) \cdot \mathbf{J}_s(\mathbf{r})dS$$
$$= \mathbf{E}^P(\mathbf{r}_0) - i\omega\mu \int_s \underline{\underline{\mathbf{G}}}(\mathbf{r}_0,\mathbf{r}) \cdot \tau(\mathbf{r})\mathbf{E}_s(\mathbf{r})dS, \quad (1)$$

where $\mathbf{E}(\mathbf{r}_0)$ is the electric field at any location $\mathbf{r}_0$, $\mathbf{E}^P(\mathbf{r}_0)$ is the primary electric field in the absence of the sheet, ω is the angular frequency, μ is the sheet's magnetic permeability which is assumed to be the same as that of the free space. $\mathbf{J}_s(\mathbf{r})$ is the scattering current in the sheet, which is the product of the tangential component of the total electric field in the sheet $\mathbf{E}_s(\mathbf{r})$ and the anomalous admittance of the sheet $\tau(\mathbf{r})$. The anomalous admittance is given by

$$\tau(\mathbf{r}) = \Delta\{\sigma(\mathbf{r}) + i\omega\varepsilon(\mathbf{r})\}\, t . \qquad (2)$$

Here, t is the thickness of the sheet, $\Delta\sigma(\mathbf{r})$ is the difference in the conductivity between the sheet and surrounding medium, and $\Delta\varepsilon(\mathbf{r})$ is the difference in the electrical permittivity. The electric dyadic Green's tensor in equation (1) satisfies

$$\nabla \times \nabla \times \underline{\underline{\mathbf{G}}}(\mathbf{r},\mathbf{r}_0) - k_j^2\, \underline{\underline{\mathbf{G}}}(\mathbf{r},\mathbf{r}_0) = \delta(\mathbf{r}-\mathbf{r}_0)\underline{\underline{\mathbf{I}}} , \qquad (3)$$

where k_j is the propagation constant for layered earth given by

$$k_j^2 = -\hat{z}_j\hat{y}_j = \omega^2\mu_j\varepsilon_j - i\omega\mu_j\sigma_j , \qquad (4)$$

and $\underline{\underline{\mathbf{I}}}$ is the identity tensor.

The expression (1) becomes the Fredholm integral equation of the second kind if the field point $\mathbf{r}_0$ is on the sheet. If we consider only the tangential components of the electric field equation (1) is reduced to

$$\mathbf{E}_s(\mathbf{r}_0) = \mathbf{E}_s^P(\mathbf{r}_0) - i\omega\mu\int_S \underline{\underline{\mathbf{G}}}_s(\mathbf{r}_0,\mathbf{r})\cdot\tau(\mathbf{r})\mathbf{E}_s(\mathbf{r})\,dS , \qquad (5)$$

where the subscript S indicates the tangential field in the sheet. The same subscript under the Green's tensor indicates the tangential electric field at $\mathbf{r}_0$ due to a scattering current source at $\mathbf{r}$. If we divide the sheet into a number of cells, the size of each of which is small enough to assume constant scattering current over the cell, then the integral in the above equation reduces to a summation of the integrals of the Green's tensor over each cell.

$$\mathbf{E}_s(\mathbf{r}_l) = \mathbf{E}_s^P(\mathbf{r}_l) - i\omega\mu\tau_n\sum_{n=1}^{N}\int_{S_n}\underline{\underline{\mathbf{G}}}_s(\mathbf{r}_l,\mathbf{r})\,dS\cdot\mathbf{E}_s(\mathbf{r}_n) , \qquad (6)$$

where N is the total number of cells. Since the sheet has only the strike a and the dip b dimensions, the Green's tensor in the sheet reduces to 2 × 2 dyadic by rotation of coordinates (Weidelt, 1981; Zhou, 1989) as

$$\underline{\underline{\mathbf{G}}}_s(\mathbf{r}_0,\mathbf{r}) = \begin{pmatrix} G_{aa} & G_{ab} \\ G_{ba} & G_{bb} \end{pmatrix} . \qquad (7)$$

For a layered earth each element of this tensor will be determined using the boundary condition which is the continuity of tangential electric and magnetic fields at layer boundaries.

The quality of numerical results depends, in part, on the accuracy of the evaluation of the Green's tensor integral over each cell. Integration over the singular cell, in which $\mathbf{r}_n = \mathbf{r}_l$ in equation (6), is of particular importance. The whole space (or, the primary part of) Green's tensor in the sheet is represented as

$$\underline{\underline{\mathbf{G}}}_s^0(\mathbf{r}_0,\mathbf{r}) = \phi^0(\mathbf{r}_0-\mathbf{r})\underline{\underline{\mathbf{I}}} + \frac{1}{k_0^2}\nabla_s\nabla_s\cdot\{\phi^0(\mathbf{r}_0-\mathbf{r})\underline{\underline{\mathbf{I}}}\} , \qquad (8)$$

where the scalar potential and differential operator are

$$\phi^0(\mathbf{r}_0-\mathbf{r}) = \frac{e^{-ik_0|\mathbf{r}_0-\mathbf{r}|}}{4\pi|\mathbf{r}_0-\mathbf{r}|} , \text{ and} \qquad (9)$$

$$\nabla_s = \frac{\partial}{\partial a}\hat{\mathbf{a}} + \frac{\partial}{\partial b}\hat{\mathbf{b}} , \qquad (10)$$

respectively. Note that the singularity exists only when integrating the whole space Green's tensor. If one uses the low induction approximation, $|k_0||\mathbf{r}_0-\mathbf{r}| << 1$, the scalar potential may reduce to that of dc. For conventional EM scattering problems with its frequency less than 100 kHz or so, the above approximation may work nicely. However, at high-frequencies where both the displacement current and the conduction current need to be considered, the low induction approximation may not be valid.

By first replacing the square singular cell with a circular disk of same area, it can be shown (Song and Lee, 1998) that the singular cell integrals have closed form results

$$\int_{S_0}\phi^0\,dS \cong \frac{i}{2k_0}\left(e^{-ik_0\rho_0} - 1\right) , \text{ and} \qquad (11)$$

$$\int_{S_0}\frac{\partial^2\phi^0}{\partial a^2}dS = \int_{S_0}\frac{\partial^2\phi^0}{\partial b^2}dS \cong -\frac{(1+ik_0\rho_0)e^{-ik_0\rho_0}}{4\rho_0} , \qquad (12)$$

where ρ_0 is the radius of the circular disk.

Separating the Green's tensor into numerically stable parts, and defining the sheets' scattering current as the sum of irrotational and solenoidal components, we generalize the approach shown by Weidelt (1981) as;

$$\underline{\underline{\mathbf{G}}}_s(\mathbf{r}_0,\mathbf{r}) = \underline{\underline{\mathbf{S}}} + \frac{1}{k_j^2}\nabla_s\Phi , \qquad (13)$$

$$\mathbf{J}_s = \nabla_s\times(\hat{\mathbf{c}}\psi) + k_j^2\nabla_s\varphi , \text{ and} \qquad (14)$$

$$\hat{\mathbf{c}} = \hat{\mathbf{a}}\times\hat{\mathbf{b}} , \qquad (15)$$

where $\underline{\underline{\mathbf{S}}}$ and $\nabla_s\Phi$ are to be determined in a layered earth using the boundary condition and the reciprocity principle. Substituting equations (13) and (14) into equation (5), one arrives at the integral equation

$$\mathbf{E}_s(\mathbf{r}_0) = \mathbf{E}_s^P(\mathbf{r}_0) - i\omega\mu\int_S\left\{\underline{\underline{\mathbf{S}}}\cdot\nabla_s\times(\hat{\mathbf{c}}\psi) + (k_j^2\underline{\underline{\mathbf{S}}} + \nabla_s\Phi)\cdot\nabla_s\varphi\right\}dS . \qquad (16)$$

This formulation is suitable for modeling EM responses of sheets over a wide range of frequencies in a layered earth of arbitrary electrical properties including the free space.

Numerical example

We have developed a numerical code HFSHEET based on the algorithm described above. The performance of the modeling code was verified by comparing its results with published ones (PLATE by Dyck, et al., 1980; Zhou, 1989) for frequencies below 100 kHz. Unfortunately, there is no published result at high frequencies in which it would be necessary to consider the displacement current. Hence we compared the results with those of EM1D (Pellerin, et al., 1995) code which provides accurate EM responses of a layered earth when excited by a dipole source anywhere in the earth.

Figure 2 shows the model used for the comparison and resulting EM fields. After numerous tests, we locate the vertical magnetic dipole source of unit moment at 2m below the surface, and observation position at 8m below surface and 2m apart from the source. The resulting EM fields shown contain the total field. The earth is composed of three layers; thin overburden with the thickness of 0.5m, 14.5m thick host and conductive basement. The resistivities of the layers are 100, 500 and 50 ohm-m from the top while the dielectric constants are 6, 1 and 10, respectively. The conductance and electric permittance of S_1 are 0.02 S and $10 \times 8.8542 \times 10^{-12}$ F, respectively, while those of S_2 are 1.0 S and $30 \times 8.8542 \times 10^{-12}$ F, respectively. In EM1D computation, the sheets were simulated with thin layers with the thickness of 0.01m. When running the HFSHEET, the horizontal extents of the sheets were 20m × 20m with cell division of 10 × 10 for frequencies lower than 10 MHz. At higher frequencies we had to use 10m × 10m sheets with cell division of a maximum of 18 × 18. The source was directly above the center of the sheets.

At low frequencies where the diffusion dominates, the EM fields mostly show the responses of the layered earth and the lower sheet S_2. As the frequency is increased, the upper sheet S_1 starts to play an increasingly more important role. Towards the high-frequency end the wave propagation dominates over the diffusion, so we can see the characteristic fluctuations as a function of frequency. Throughout the entire frequency range HFSHEET results match very well with the EM1D result.

Conclusion

We have developed a wide-band thin sheet EM modeling scheme using the integral equation method. Currently, it can handle two sheets of arbitrary properties. The performance of the resulting code HFSHEET has been verified over a wide range of frequencies. For accurate evaluation of the Hankel transform involved in the computation of Green's functions, Gaussian quadrature scheme is used instead of the usual digital Hankel transform. Further numerical accuracy was achieved by considering the branch point in carrying out the Gaussian quadrature.

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References

Anan, A. P., 1974, The equivalent source method for electromagnetic scattering analysis and its geophysical application: Ph.D. thesis, Memorial Univ. of Newfoundland.

Cole, K. S., and Cole, R. H., 1941, Dispersion and adsorption in dielectrics: J. Chem. Phys., **9**, 341.

Dyck, A. V., Bloore, M, and Vallee, M. A., 1980, User manual for programs PLATE and SPHERE: Res. In Applied Geophysics, **14**, Univ. of Toronto.

Fainberg, E. B., Pankratov, O. V., and Singer, R. Sh., 1993, Thin sheet modelling of subsurface and deep inhomogeneities: Geophys. J. Int., **113**, 144-154.

Joshi, M. S., Gupta, O. P., and Negi, J., G., 1988, On the effects of thickness of the half-plane model in HLEM induction prospecting over sulphide dykes in a highly resistive medium: Geophys. Prosp., **36**, 551-558.

Lajoie, J. J., and West, G. F., 1976, The electromagnetic response of a conductive inhomogeneity in a layered earth: Geophysics, **41**, 1133-1156.

Pellerin, L., Labson, V.F., and Pfeifer, M.C., 1995, VETEM - A very early time electromagnetic system -, Proceeding of the Symposium on the Application of Geophysics to Engineering and Environmental Problems (SAGEEP), 725-731.

Price, A. T., 1949, The induction of electric currents in non-uniform thin sheets and shells: Quart. J. Mech, and App. Math., **2**, 283-310.

Song, Y., and Lee, K.H., 1998, Analytic singular cell integral of Green's tensor in thin sheet EM modeling, in preparation for submission to Geophysics.

Walker, P. W., and West, G. W., 1991, A robust integral equation solution for electromagnetic scattering by a thin plate in conductive media: Geophysics, **56**, 1140-1152.

Weidelt, P., 1981, Report on dipole induction by a thin plate in a conductive halfspace with an overburden: Fed. Inst. Earth Sci. and Mate.

Zhou, Q., 1989, Audio-frequency electromagnetic tomography for reservoir evaluation: Ph.D. thesis, Univ. of California.

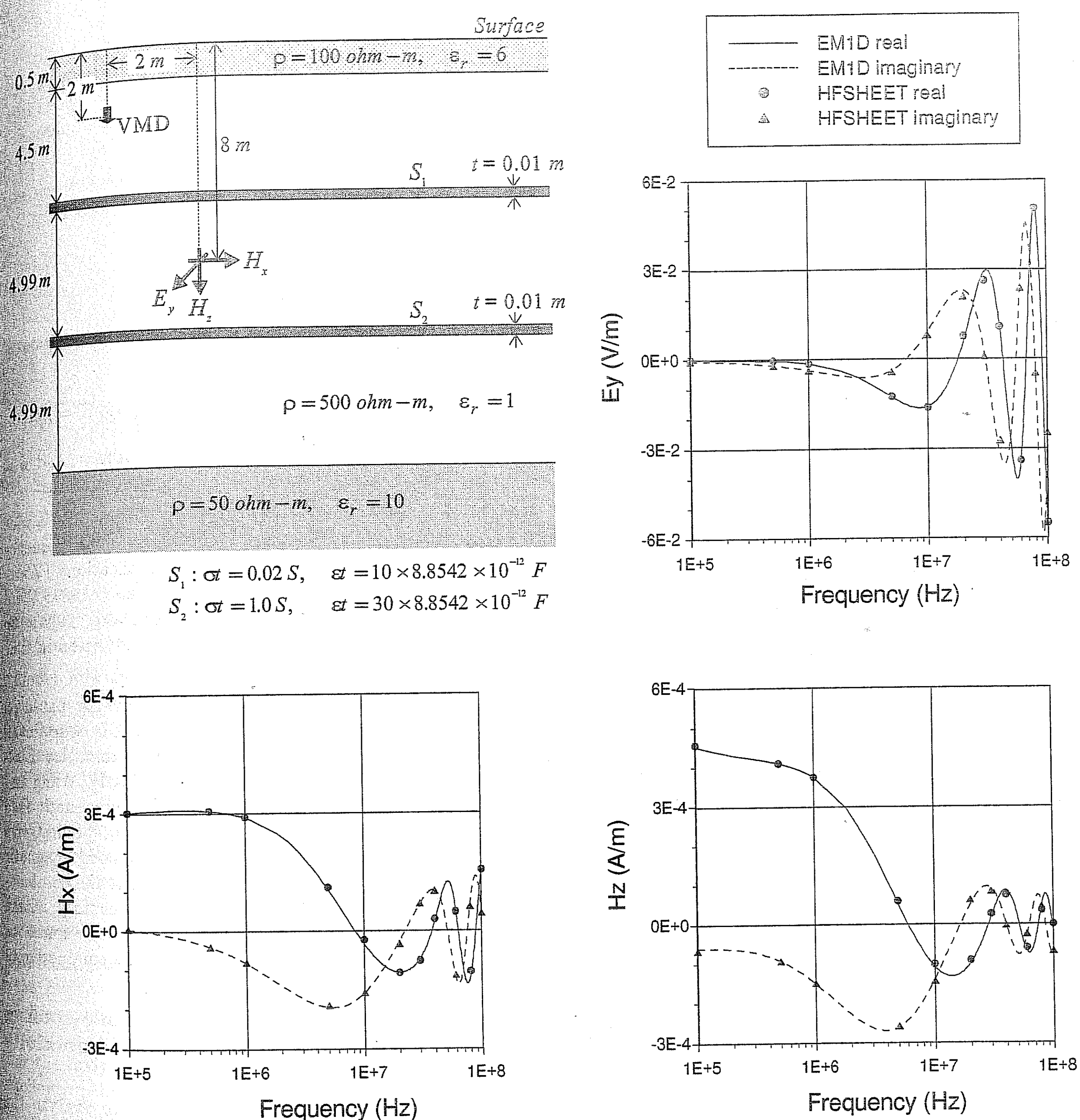

Fig. 2 Comparison between EM1D and HFSHEET. Upper left shows the model including two thin conductive sheets embedded in a three-layered earth (not exact in scale). In EM1D, the sheets were simulated with thin layers with thickness of 0.01 m. For HFSHEET, the size of each sheet was 20 m × 20 m for frequencies below 10 MHz, divided by 10 × 10 cells, while it was 10 m × 10 m above 10 MHz, divided by maximum 18 × 18 cells. The vertical magnetic dipole source of unit moment was located above the center of the sheets.